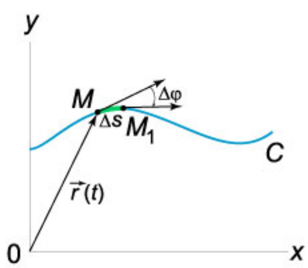
Exercise 1.2.31, page 61 curvature in parametric form
from *Introduction to Tensor Calculus* by J.H. Heinbockel

Show that when the equation of a curve is given in the parametric form $x = x(t)$ and $y = y(t)$, then the curvature $\kappa = \pm\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}$ and remains invariant under the change of parameter $t = t(\bar{t})$, where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$, etc.

Proof

The curvature of the curve can be defined as the ratio of the rotation angle of the tangent $\triangle\phi$ to the traversed arc length $\triangle s$. This ratio $\frac{\triangle\phi}{\triangle s}$ is called the average curvature of the curve. When the point M1 tends to the point M, we obtain the curvature of the curve at the point M:
$\kappa = \lim\limits_{\triangle s \to 0} \frac{\triangle\phi}{\triangle s}$.
It is clear that the curvature κ in the general case can be either positive or negative, depending on the direction of rotation of the tangent.



By definition curvature $\kappa = \frac{d\phi}{ds}$, where ϕ is the tangential angle and s is the arc length.
Thus $\kappa = \frac{d\phi}{ds} = \frac{\frac{d\phi}{dt}}{\frac{ds}{dt}} = \frac{\frac{d\phi}{dt}}{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}} = \frac{\frac{d\phi}{dt}}{\sqrt{\dot{x}^2+\dot{y}^2}}$.
Since $\tan\phi = \frac{dy}{dx}$, then $\tan\phi = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}}$,
so $\frac{d(\tan\phi)}{dt} = \frac{d\left(\frac{\dot{y}}{\dot{x}}\right)}{dt} = \frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^2}$.
Also $\frac{d(\tan\phi)}{dt} = \sec^2\phi\frac{d\phi}{dt} = \left(1+\tan^2\phi\right)\frac{d\phi}{dt}$. So
$\frac{d\phi}{dt} = \frac{1}{1+\tan^2\phi}\frac{d(\tan\phi)}{dt} = \frac{1}{1+\left(\frac{\dot{y}}{\dot{x}}\right)^2}\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^2} = \left(\frac{\dot{x}^2}{\dot{x}^2+\dot{y}^2}\right)\left(\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^2}\right) = \frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^2+\dot{y}^2}$.

Thus $\kappa = \frac{\frac{d\phi}{dt}}{\frac{ds}{dt}} = \frac{\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^2+\dot{y}^2}}{\pm\sqrt{\dot{x}^2+\dot{y}^2}} = \pm\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}$. QED

Suppose $t = t(\bar{t})$, then $\tan\phi = \frac{dy}{dx} = \frac{\frac{dy}{dt}\frac{dt}{d\bar{t}}}{\frac{dx}{dt}\frac{dt}{d\bar{t}}} = \frac{\dot{y}\frac{dt}{d\bar{t}}}{\dot{x}\frac{dt}{d\bar{t}}} = \frac{\dot{y}}{\dot{x}}$.
Thus $\kappa = \frac{\frac{d\phi}{d\bar{t}}}{\frac{ds}{d\bar{t}}} = \frac{\frac{d\phi}{dt}\frac{dt}{d\bar{t}}}{\frac{ds}{dt}\frac{dt}{d\bar{t}}} = \frac{\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^2+\dot{y}^2}\frac{dt}{d\bar{t}}}{\pm\sqrt{\dot{x}^2+\dot{y}^2}\frac{dt}{d\bar{t}}} = \pm\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}$.
So κ is invariant with respect to t. QED